ON CARDINAL COLLAPSING WITH REALS

BY

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ABSTRACT

We show that every uncountable regular cardinal can become $\aleph_1$ in a suitable Cohen extension by a real (set of integers) without destroying larger cardinals. The proof is analyzed to obtain results about the powers of Boolean algebras.

The following result, which is the main theorem of this paper, is almost proved by Jensen and Solovay in [4, Sect. 3].

THEOREM 1. *Let M be a countable transitive $\in$-model of* ZFC, *and let θ be a regular uncountable cardinal in M. Then there is an $a \subseteq \omega$ such that $M[a]$ is a Cohen extension of M having the same cardinals greater than or equal to θ as M and satisfying $\aleph_1^{M[a]} = \theta$.*

We prove the theorem in Section 1 by a small modification of the argument of [4, (3.3)–(3.6)]. In Section 2 we analyze the construction in terms of Boolean algebras, and in Section 3 we determine the possible non-strongly Mahlo powers of countably generated complete Boolean algebras.

We assume knowledge of [4, Sect. 2–3], and follow its terminology.

1. Proof of the Theorem 1

Let M be a fixed, countable, transitive $\in$-model of ZFC, θ an uncountable regular cardinal in M. We wish to prove that

(1) there is an $a \subseteq \omega$ such that $M[a]$ is a model of ZFC, $\aleph_1^{M[a]} = \theta$, and $M[a]$ has the same cardinals greater than or equal to θ as M.

The only additional claim in Theorem 1 is that $M[a]$ is a Cohen extension of M. Here we prove only (1). In Section 2 a simple analysis of the proof will show that we actually have a Cohen extension.

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LEMMA 2. *(1) holds if* $\theta \leqq (2^{\aleph_0})^M$.

PROOF .(cf. [4, (3.3)–(3.6)]). Let $A \subseteq \theta$ be such that $M[A]$ is a Cohen extension of M, $\aleph_1^{M[A]} = \theta$ and $M[A]$ has the same cardinals greater than or equal to θ as M (see [4, (3.3)]). Next let W_1 $(\in M)$, W_2 $(\in M[A])$ be well-orderings of the reals (subsets of ω) in M and $M[A]$ respectively. We define the sequence of reals $\langle a_\xi \mid \xi < \theta \rangle$ in $M[A]$ as follows.

If $\xi < \theta$ and ξ is not a limit ordinal, a_ξ is the first real in $M[A]$ (under W_2) which codes the ordinal ξ (that is, which has the form $\{2^i 3^j \mid (i,j) \in R\}$ where R is some ordering of integers in order type ξ).

If $\xi < \theta$ and ξ is a limit ordinal, a_ξ is the first real in M (under W_1) which does not code any ordinal and is distinct from a_η for all $\eta < \xi$.

Before proceeding we must show that for each $\xi < \theta$, such a real a_ξ exists. For nonlimit ξ this is obvious since every $\xi < \theta$ is countable in $M[A]$. For limit ξ note the following: the set $E = \{x \in M \mid x$ is a real which does not code an ordinal$\}$ is in M, and has power $2^{\aleph_0}$ in M (since $E \supseteq \{x \in M \mid 5 \in x \subseteq \omega\}$). Since $\theta \leqq (2^{\aleph_0})^M$, there is in M a one-to-one function $f\colon \theta \to E$. Now suppose $\xi < \theta$ is limit and a_η exists for all $\eta < \xi$. If a_ξ does not exist this means $E \subseteq \{a_\eta \mid \eta < \xi\}$ and so, since $\langle a_\eta \mid \eta < \xi \rangle \in M[A]$ we have in $M[A]$ a one-to-one function $g\colon E \to \xi$. Combining f and g we have in $M[A]$ a one-to-one function $h\colon \theta \to \xi$, contradicting the fact that $\xi < \theta = \aleph_1^{M[A]}$.

To sum up, we have in $M[A]$ a sequence of reals $\langle a_\xi \mid \xi < \theta \rangle$ without repetitions, such that for nonlimit $\xi < \theta$, a_ξ codes ξ.

We now follow word for word [4, (3.5)] to obtain a real a (called x in [4]) such that $M[A, a]$ is a Cohen extension of $M[A]$ with the same cardinals as $M[A]$, and $R(a, \varnothing) = a_0$, $R(a, a_\xi) = a_{\xi+1}$ for all $\xi < \theta$. (There is no need to recall how the operation R on reals is defined; we only need to know that its definition is absolute for transitive ε-models of ZFC.)

It is now clear that $M[a]$ is a model of ZFC. The sequence $\langle a_\xi \mid \xi < \theta \rangle$ is in $M[a]$ because it can be defined within $M[a]$ by induction as follows:

$a_0 = R(a, \varnothing)$, $a_{\xi+1} = R(a, a_\xi)$ and for limit ξ one repeats the original definition of a_ξ.

Therefore every nonlimit $\xi < \theta$ is countable in $M[a]$, so $\theta \leqq \aleph_1^{M[a]}$. But $\theta = \aleph_1^{M[A,a]}$ hence $\theta = \aleph_1^{M[a]}$. Cardinals of M greater than θ are cardinals in $M[A, a]$ hence in $M[a]$. This completes the proof of (1), assuming $\theta \leqq (2^{\aleph_0})^M$.

LEMMA 3. *(1) holds if θ is not strongly inaccessible in M.*

PROOF. We are assuming that θ is regular and uncountable. If in addition, θ is not strongly inaccessible, choose an infinite cardinal λ of M such that $\lambda < \theta \leqq (2^\lambda)^M$. Let x be a real such that $M[x]$ is a Cohen extension of M in which λ is countable and every $\mu > \lambda$ which is a (regular) cardinal in M is a (regular) cardinal in $M[x]$ (refer to [4, (3.2), Rem. 2]). Since $\theta > \lambda$ and θ is regular in M, θ is uncountable and regular in $M[x]$. Also, clearly, $M[x] \models \theta \leqq 2^{\aleph_0}$ (since in M there is a one-to-one function from θ into subsets of λ). Thus Lemma 2 is applicable to $M[x]$ and θ, and yields the existence of some $y \subseteq \omega$ such that $M[x, y]$ is a model of ZFC in which $\theta = \aleph_1$ and which has the same cardinals $\geqq \theta$ as $M[x]$ (hence as M). Using a standard pairing function of reals we obtain a real a such that $M[a] = M[x, y]$, and clearly $M[a]$ has all the properties required in (1).

LEMMA 4. (1) *holds if* θ *is strongly inaccessible in* M.

This is part of [4, Th. 3.2].

Combining Lemmas 3 and 4 we have the desired result.

2. Construction of the associated Boolean algebra

We still have to show that $M[a]$ of Section 1 is a Cohen extension of M (that is, it is obtained from M by adjoining an M-generic filter on some partially ordered set in M). A look at the proof in Section 1, and the proof of [4, Th. 3.2] (which we needed for Lemma 4) shows the model $M[a]$ is always obtained from M by a sequence of steps, each of which is the formation of a Cohen extension or of a submodel (for example, $M[a] \subseteq M[A, a]$), so that the assertion follows immediately from the following two known claims.

Claim (I) If M' is a Cohen extension of M and M'' a Cohen extension of M', then M'' is a Cohen extension of M.

Claim (II). If N is a Cohen extension of M, $a \in N$, $a \subseteq M$, then $M[a]$ is a Cohen extension of M.

Once one knows the equivalence of extensions by generic filters on partially ordered sets with extensions by generic ultrafilters on complete Boolean algebras (henceforth referred to as CBAs) (see [2, Lemma 45, p. 52]), Claims I and II reduce to the corresponding results about ultrafilter extensions, which follow easily from [2, Lem. 85, p. 100 and Lem. 69, p. 86, respectively]. (The original source is [6], but the formulations of Jech are just what we need at the moment.)

This completes the proof of Theorem 1. To analyze the Boolean algebraic implications, it is slightly more convenient now to leave the countable models and adopt the framework of Boolean-valued models in which Solovay and Tennenbaum work in [6]. We shall assume the properties of $V^{(\mathscr{B})}$ ($\mathscr{B}$ any CBA) reviewed or proved in [6, Sects. 3–5]; we write $V^{(\mathscr{B})} \models \cdots$ for $\| \cdots \|^{(\mathscr{B})} = 1$. Two facts which are mentioned in [6] in connection with the countable chain condition (CCC) will be needed below in the more general case of CBAs satisfying the θ chain condition;

(2) there are no θ disjoint nonzero elements,

where θ is any uncountable regular cardinal. The first is that if $\mathscr{B}$ is a CBA satisfying (2) then for every ordinal $\alpha \geqq \theta$

$$\alpha \text{ is a (regular) cardinal} \Rightarrow V^{(\mathscr{B})} \models \hat{\alpha} \text{ is a (regular) cardinal.}$$

This is well known. (Remark: We use $\hat{\alpha}$ to denote the canonical embedding of V in $V^{(\mathscr{B})}$. For typographical reasons we are unable to use ˇ as appears in [6].)

The second fact is that [6, Lem. 5.2.6, p. 215] generalizes to (2) as follows.

Let $\mathscr{B}$ be a CBA, $\mathscr{D}$ a CBA in $V^{(\mathscr{B})}$, $\mathscr{C} = \mathscr{B} \mathbin{\tilde{\otimes}} \mathscr{D}$. Then, for any uncountable regular cardinal θ, the following are equivalent:

(i) $\mathscr{C}$ satisfies (2).

(ii) $\mathscr{B}$ satisfies (2) (so that $V^{(\mathscr{B})} \models \hat{\theta}$ is a regular cardinal) and $V^{(\mathscr{B})} \models$ $\mathscr{D}$ satisfies

(2′) the $\hat{\theta}$ chain condition.

The proof of this for the case $\theta = \aleph_1$, given in [6, pp. 223–224] generalizes almost word for word.

The analogue of Theorem 1 in the present framework is Theorem 5.

THEOREM 5. *Let θ be an uncountable regular cardinal. There is a countably generated* CBA $\mathscr{B}$ *such that* $V^{(\mathscr{B})} \models \hat{\theta} = \aleph_1$, *and for every cardinal* $\kappa \geqq \theta$, $V^{(\mathscr{B})} \models \hat{\kappa}$ *is a cardinal. Moreover, if θ is not strongly Mahlo, then $\mathscr{B}$ satisfies* (2), *and in every case $\mathscr{B}$ satisfies*

(2″) *the θ^+ chain condition.*

Let us sketch the proof for the case $\theta \leqq 2^{\aleph_0}$, corresponding to Lemma 2. One begins with the set $\mathscr{P}_1$ of forcing conditions needed to collapse all cardinals below θ to $\aleph_0$ (see [5, pp. 14–15] for the definition of $\mathscr{P}_1$ and the proof that there are no θ pairwise-incompatible conditions). Let $\mathscr{B}_1$ be the CBA associated with $\mathscr{P}_1$ as in [6, (7.6)]. (We denote $\mathscr{B}_1 = \mathrm{RO}(\mathscr{P}_1)$, where RO is for regular open.) $\mathscr{B}_1$ satisfies (2) and $V^{(\mathscr{B}_1)} \models \hat{\theta} = \aleph_1$.

We now repeat in $V^{(\mathscr{B}_1)}$ what was done in $M[A]$ in the proof of Lemma 2. In view of the maximum principle [6, (3.9), 1], there is some $f \in V^{(\mathscr{B}_1)}$ such that, letting X be the set of all reals and W a well-ordering of it, the following holds in $V^{(\mathscr{B}_1)}$:

(3) f is a sequence of reals of length $\hat{\theta}$; f is one-to-one; for nonlimit $\xi < \hat{\theta}, f(\xi)$ is a code for ξ; for limit $\xi < \hat{\theta}$, $f(\xi)$ is the first element of $\hat{X}$, under $\hat{W}$, which does not code an ordinal and differs from $f(\eta)$ for all $\eta < \xi$.

Now, the argument of [4, (2.4)–(2.5), (3.5)] (that is, the trick of almost-disjoint sets) can be viewed as the proof of Theorem 6. (The operation R is defined in [4, (3.5)]; again, the specific definition has no importance once one notes that it is absolute.)

THEOREM 6. *Let f be a sequence without repetition of reals, whose domain is a limit ordinal α. Then there is a CBA $\mathscr{D}$ satisfying the* CCC *such that the following holds in $V^{(\mathscr{D})}$: $(\exists x \subseteq \omega)$ $[R(x, \varnothing) = \hat{f}(0)$ and for all $\xi < \hat{\alpha}$, $R(x, \hat{f}(\xi)) = \hat{f}(\xi + 1)]$.*

Since Theorem 6 is a theorem of ZFC it holds in $V^{(\mathscr{B}_1)}$. Using it there for the element $f \in V^{(\mathscr{B}_1)}$ chosen earlier and for $\alpha = \hat{\theta}$, we obtain by the maximum principle an element $\mathscr{D} \in V^{(\mathscr{B}_1)}$ such that $V^{(\mathscr{B}_1)} \models [\mathscr{D}$ is a CBA satisfying the CCC and $\| (\exists x \subseteq \omega\ [R(x,\varnothing) = \hat{f}(0)$ and for all $\xi < \hat{\theta}$, $R(x, \hat{f}(\xi)) = \hat{f}(\xi + 1)] \|^{[\mathscr{D}]} = 1]$.

Here we obviously have the situation for which the results of [6, Sect. 5] have been developed. So let us form the CBA $\mathscr{C} = \mathscr{B}_1 \tilde{\otimes} \mathscr{D}$. Since $\mathscr{B}_1$ satisfies (2) and $\mathscr{D}$ satisfies (2′) in $V^{(\mathscr{B}_1)}$, $\mathscr{C}$ satisfies (2). The isomorphism between (what might be denoted) $V^{(\mathscr{B})^{(\mathscr{D})}}$ and $V^{(\mathscr{C})}$ leads to the following result. (To be precise, we are using [6, Lem. 5.3.3].)

$$\| (\exists x \subseteq \omega) [R(x, \phi) = f(0) \text{ and for all } \xi < \hat{\theta},\ R(x, f(\xi)) = f(\xi + 1)] \|^{(\mathscr{C})} = 1.$$

Actually we should write here $i_*(f)$ rather than f here, where i_* is the natural embedding of $V^{(\mathscr{B}_1)}$ in $V^{(\mathscr{C})}$. However, we identify $V^{(\mathscr{B}_1)}$ with its image in $V^{(\mathscr{C})}$, and may also regard $V^{(\mathscr{B}_1)}$ as a class in $V^{(\mathscr{C})}$, so that conditions (3) above, which are absolute, hold also in $V^{(\mathscr{C})}$. (The formal machinery needed in this step is given in [6, (3.6)].)

Using the maximum principle in $V^{(\mathscr{C})}$ choose $a \in V^{(\mathscr{C})}$ such that

$$V^{(\mathscr{C})} \models [a \subseteq \omega,\ R(a, \varnothing) = f(0) \text{ and for all } \xi < \hat{\theta},\ R(a, f(\xi)) = f(\xi + 1)].$$

Now let $\mathscr{B}$ be the complete subalgebra of $\mathscr{C}$ generated by $\{ \| \hat{n} \in a \|^{(\mathscr{C})} \mid n < \omega \}$.

$\mathscr{B}$ is countably generated and satisfies (2). Again, we regard $V^{(\mathscr{B})}$ as a class in $V^{(\mathscr{C})}$. This class contains all standard elements ($\hat{x}$ for all x) as well as a, and is a transitive model of ZFC. Therefore it contains f (by the same reasoning which showed in Section 1 that $\langle a_\xi \mid \xi < \theta \rangle \in M[a]$; the absoluteness of R is needed here) and f satisfies (3) in it. It follows that this class satisfies $\hat{\theta} = \aleph_1$ and has the same cardinals $\geqq \hat{\theta}$ as V and $V^{(\mathscr{C})}$. (To be precise, we should have talked in $V^{(\mathscr{C})}$ about the class $T^{(\mathscr{B})}$, in the notation of [6, (3.6)].) Therefore $V^{(\mathscr{B})}$ has all the desired properties, and we have completed the proof of Theorem 5 for $\theta \leqq 2^{\aleph_1}$.

The treatment of the case where θ is not strongly inaccessible, corresponding to Lemma 3, is similar and even easier. Again one combines two Boolean extensions which satisfy (2) (the first makes $\hat{\theta} \leqq 2^{\aleph_0}$ and the second, given by Theorem 5 for the case $\theta \leqq 2^{\aleph_0}$, makes $\hat{\theta} = \aleph_1$) and so the product CBA $\mathscr{C}$ satisfies (2). There is no need to take a subalgebra, since $\mathscr{C}$ can be seen to be countably generated by using [6, Lem. 5.2.5].

We now recall the definition of Mahlo numbers. An infinite cardinal θ is called weakly Mahlo when every closed and unbounded subset of θ contains a regular cardinal (this implies that θ is weakly inaccessible, and is equivalent to the definition of [4, (3.1)]). θ is called strongly Mahlo when every closed and unbounded subset of θ contains a strongly inaccessible cardinal (this implies that θ is strongly inaccessible). Note that if θ is strongly inaccessible, A is a closed and unbounded subset of θ and $B = \{\alpha \in A \mid \alpha \text{ is a limit cardinal, and for all } \kappa < \alpha,\ 2^\kappa < \alpha\}$, then B is closed and unbounded too, and every regular cardinal in B is strongly inaccessible. This substantiates Lemma 7.

LEMMA 7. *θ is strongly Mahlo iff it is strongly inaccessible and weakly Mahlo.*

Now, Solovay's proof of [4, (3.2)] for θ not weakly Mahlo is very similar to our proof of Lemma 2 (in fact, it served as the model for our proof) and it can be analyzed in the same way to obtain a countably generated CBA $\mathscr{B}$ satisfying (2) and $V^{(\mathscr{B})} \models \hat{\theta} = \aleph_1$. Combining this with our previous result and Lemma 7 the proof of Theorem 5 is complete for θ not strongly Mahlo.

If θ is strongly Mahlo one needs Jensen's argument [4, (3.7)–(3.9)]. The desired CBA $\mathscr{B}$ appears as a countably generated subalgebra of the CBA, $\mathscr{C}$ corresponding to a three-stage extension $M \mapsto M[f] \mapsto M[f, A] \mapsto M[f, A, x]$ in the notation of [4, (3.7)]. The set of forcing conditions in the first step has power θ ([4, end of (3.9)]) and thus the corresponding CBA satisfies (2″). In the second and third step even (2′) is satisfied, hence the product algebra $\mathscr{C}$ satisfies (2″), and so does its subalgebra $\mathscr{B}$. This completes the proof of Theorem 5.

REMARK. Here is an alternative approach to Theorem 5. If M is a transitive $\in$-model of ZFC, θ a regular cardinal in M, then by a θ-extension of M we mean a generic ultrafilter-extension of M through a CBA satisfying (2) (equivalently a Cohen extension through a partially ordered set satisfying the θ antichain condition). In analogy with Claims I, II above, it is not hard to prove the following.

Claim (I)′. If M' is a θ-extension of M and M'' is a θ-extension of M' then M'' is a θ-extension of M.

Claim (II)′. If N is a θ-extension of M, $a \in N$, $a \subseteq \omega$, then $M[a]$ is a θ-extension of M through a countably generated CBA.

These claims and the proof of our Theorem 1 imply rather easily the truth of Theorem 5 in M, hence, by well-known arguments, in the universe.

3. The powers of countably generated CBAs

This paper developed from the attempt to determine exactly the possible powers of countably generated CBAs. Some basic results in this direction are reported in [7, Sect. 3] and here we shall prove Theorem 8.

THEOREM 8. *Let ν be a cardinal which is not strongly Mahlo. ν is the power of a countably generated CBA iff $\nu = 2^n$ for some $n < \omega$ or $\nu = 2^{\underset{\smile}{\theta}}$ for some regular uncountable cardinale θ where $2^{\underset{\smile}{\theta}} = \sum_{\lambda<\theta} 2^{\lambda}$.*

PROOF. We ignore the trivial case of finite ν. Let $\nu = |\mathscr{B}|$ where $\mathscr{B}$ is an infinite countably generated CBA and let θ be the smallest cardinal (denoted by $\mathrm{CC}(\mathscr{B})$) such that $\mathscr{B}$ satisfies (2). By [1, Th. 1] θ is regular and greater than $\aleph_0$. By [7, (9.3)] (which the reader can probably prove for himself), $|\mathscr{B}| = 2^{\underset{\smile}{\mathrm{CC}(\mathscr{B})}}$ so that $\nu = 2^{\theta}$. This proves the $\rightarrow$ direction of the iff in Theorem 6. For the $\leftarrow$ direction we use the assumption that ν is not strongly Mahlo. Suppose $\nu = 2^{\underset{\smile}{\theta}}$ where θ is regular and greater than $\aleph_0$. Clearly θ is not strongly Mahlo. If $\theta = \aleph_1$, ν is the power of the CBA of all subsets of ω. If $\theta > \aleph_1$, let $\mathscr{B}$ be a CBA as in Theorem 5. $\mathscr{B}$ satisfies (2) but not the less than κ chain condition for any $\kappa < \theta$ (for this would imply, if $\kappa > \aleph_0$, that $V^{(\mathscr{B})} \models \aleph_1 \leqq \kappa < \theta = \aleph_1$). Thus $\theta = \mathrm{CC}(\mathscr{B})$ and by [7, (9.3)] again $|\mathscr{B}| = 2^{\underset{\smile}{\theta}} = \nu$. This completes the proof.

REMARK. In the same way, using notation and results of [7, (9.1)(2)] we can prove the following statement.

(4) Let λ, μ be infinite cardinals, λ not strongly Mahlo. Then λ is the power of some $(\aleph_0, <\mu)$-generated CBA iff $\lambda = 2^{\underset{\smile}{\theta}}$ for some regular uncountable cardinal $\theta \leqq \mu$.

Now let λ be strongly Mahlo. Is there a countably generated CBA $\mathscr{B}$ such that $|\mathscr{B}| = \lambda$ (equivalently $CC(\mathscr{B}) = \lambda$)? The question is open except for the case of weakly compact λ, in which the negative solution was discovered independently by M. Magidor, T. J. Jech, and K. Kunen (and perhaps others). In fact, the argument of [3, Sect. 3] shows that if $\mathscr{B}$ is any CBA of the weakly compact power κ, and A is a subset of $\mathscr{B}$, $|A| < \kappa$, then $\mathscr{B}$ has a complete proper subalgebra $\mathscr{C}$ containing A. Hence, as Jech notes, $\mathscr{B}$ is not generated by any set of power less than κ.

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